SUBELLIPTIC HARMONIC MAPS

JÜRGEN JOST AND CHAO-JIANG XU

ABSTRACT. We study a nonlinear harmonic map type system of subelliptic PDE. In particular, we solve the Dirichlet problem with image contained in a convex ball.

1. INTRODUCTION

Building upon and extending the basic work of Hörmander [H], a general theory of subelliptic PDE has been developed in the work of several authors (see e.g. [B], [FP], [Je], [JeS], [NSW], [RS], [S], [X1], [X2] and many others), and it has been found that many important results carry over from the elliptic to the hypoelliptic case. It therefore seems natural to try to extend also the recent advances in elliptic PDE concerned with quasilinear and nonlinear problems of a variational origin to the hypoelliptic case. The present paper aims to take a first step in this direction and to establish some foundations by investigating in detail a variational problem that turned out to be a model problem for the elliptic case, namely the harmonic map problem, in a hypoelliptic setting. One of the important features of the harmonic map problem is its geometric nature, so that geometric intuition can guide the development of analytic tools. From a geometric point of view, however, hypoelliptic operators lead to a highly singular domain metric. Fortunately, we have here the recent work of the first author ([J2], [J3], [J4]) at our disposal; he constructed a variational harmonic map theory for very general singular domains and also dissociated the underlying integration measure from the domain metric, which is required by the present context as well.

After the original work of Eells-Sampson [ES], Al′ber [A1], [A2] and Hamilton [H] on harmonic maps with values in Riemannian manifolds of nonpositive curvature that employed parabolic methods, Hildebrandt-Kaul-Widman [HKW] in a fundamental paper introduced variational methods that also allowed a treatment of positive image curvature. Here, however, one meets a principal limitation of the harmonic map theory, because Hildebrandt-Kaul-Widman discovered that harmonic maps may develop singularities unless their range satisfies a certain convexity condition. They then established an existence and regularity result under this convexity condition.

In the present paper, we first collect some results about hypoelliptic PDE from the literature, in particular about the behavior of the associated Green functions. We then use the constructions of [J1], [J2] to define an energy integral which we then

attempt to minimize. The main result of our paper is an existence and regularity theorem for harmonic maps under the convexity condition of [HKW] on the image. The proof will in particular depend on precise Green function estimates. The somewhat complicated iteration technique of [HKW] and [HW] will be avoided by utilizing a more geometric construction of Kendall [K].

In order to place the present work into a proper perspective, apart from the relations to hypoelliptic PDEs, we should also mention the relation to the regularity aspects of generalized harmonic maps as further explored in [J5]. Finally, we hope that this direction of research will also be useful for questions on the rigidity of group representations, like discrete groups acting on spaces that carry a natural Hörmander type geometry.

We shall employ standard summation conventions.

2. The geometry defined by vector fields satisfying Hörmander's condition

Let Ω be an open domain in $\mathbb{R}^n$, $n \geq 2$. Let $X_1, \ldots, X_m$ be a system of vector fields with smooth real coefficients defined on some $\tilde{\Omega}$ with $\Omega \subset\subset \tilde{\Omega}$. The essential hypothesis is Hörmander's condition:

(H): $X_1, \ldots, X_m$ together with their commutators up to a certain fixed length r span the tangent space at each point of $\tilde{\Omega}$.

For Ω, we suppose that $\partial\Omega$ is smooth and noncharacteristic for the system $X_1, \ldots, X_m$, i.e. $\forall x_0 \in \partial\Omega \ \exists j_0$ such that $X_{j_0}(x_0) \notin T_{x_0}\partial\Omega$. In general, our system of vector fields is degenerate elliptic.

Example. $X_1 = \partial_{x_1}$, $X_2 = \partial_{x_2} + x_1^k \partial_{x_3}$, $k \geq 1$, is a Hörmander type system on $\mathbb{R}^3$ with $r = k + 1$.

For abbreviation, we shall occasionally write $X := (X_1, \ldots, X_m)$. From this system of vector fields, we can define a non-isotropic metric in the following way. For $0 < \delta < 1$, we say $\varphi : [0, 1] \to \tilde{\Omega}$ belongs to $C(\delta)$ if φ is Lipschitz and

$$\dot{\varphi}(t) = \sum_{j=1}^{m} \alpha_j(t) X_j(\varphi(t)), \quad \text{a.e. } t \in [0, 1],$$

with $|\alpha_j(t)| < \delta$. We define for $x, y \in \Omega$, $|x - y| \ll 1$,

$$\varrho(x, y) := \inf\{\delta, \exists \varphi \in C(\delta), \varphi(0) = x, \varphi(1) = y\}.$$

The Hörmander condition implies that

$$(2.1) \qquad \varrho(x, y) \leq \varrho(x, z) + \varrho(z, y) \quad \forall x, y, z \in \Omega,$$

$$(2.2) \qquad C_1 |x - y| \leq \varrho(x, y) \leq C_2 |x - y|^{\frac{1}{r}}$$

for all $x, y \in \tilde{\Omega}$, $|x - y| \ll 1$ (see [Je], [JeS]). We write $X_j = b^{jk}\partial_{x^k}$; the adjoint of X_j then is defined by $X_j^* f = -\partial_{x^k}(b^{jk}f)$ for $f \in C_0^1(\tilde{\Omega})$. We consider the Hörmander operator

$$(2.3) \qquad H = \sum_{j=1}^{m} X_j^* X_j = -\sum_{i,k=1}^{m} \partial_{x^i}\left(a^{ik}(x)\partial_{x^k}\right)$$

with $a^{ik}(x) = b^{ji}(x)b^{jk}(x)$. In particular, $a^{ik} \in C^\infty(\tilde{\Omega})$ and $(a^{ik}(x))_{i,k=1,\ldots,m}$ is symmetric and positive semidefinite. It need not be positive definite, however, and

therefore, in general, H is not elliptic. We approximate H by the second order elliptic operator

$$H_\varepsilon = H - \varepsilon\Delta \quad (\varepsilon > 0).$$

These operators are Laplace type operators associated with the Riemannian metrics

$$(a_{ik}^\varepsilon(x)) = \left(a^{ik}(x) + \varepsilon\delta_{ik}\right)^{-1}$$

on $\tilde{\Omega}$. For the associated distance functions $d_\varepsilon(\cdot,\cdot)$, we have (see [S])

$$d_\varepsilon(x,y) \leq d_{\varepsilon'}(x,y) \quad \text{for } 0 < \varepsilon' \leq \varepsilon$$

and

(2.4) $$\lim_{\varepsilon\to 0} d_\varepsilon(x,y) = \varrho(x,y).$$

For the associate balls $B_H(x,\delta) = \{y; \varrho(x,y) < \delta\}$, we have (see [Je])

(2.5) $$|B_H(x,2\delta)| \leq C\,|B_H(x,\delta)|.$$

(Ω, ϱ) is some kind of degenerate Riemannian manifold because the distance balls satisfy

$$B_H(x,\delta) = \bigcap_{\varepsilon>0} B_{H_\varepsilon}(x,\delta).$$

In the theory of subelliptic PDE, such an (Ω, ϱ) is called a homogeneous space.

We shall also need the following construction: From the above description of ϱ, one concludes that

(2.6) $$|X_j\varrho| \leq 1 \quad \text{for all } j.$$

If now $0 < R_1 < R_2$ and $B_H(x,R_2) \subset \Omega$, we put

$$\eta(y) := \begin{cases} 1 & \text{for } y \in B_H(x,R_1) \\ 0 & \text{for } y \in \Omega \setminus B_H(x,R_2) \\ \frac{R_2 - \varrho(x,y)}{R_2 - R_1} & \text{for } y \in B_H(x,R_2) \setminus B_H(x,R_1). \end{cases}$$

Then η is a Lipschitz function with $0 \leq \eta \leq 1$, and

(2.7) $$|X_j\eta| \leq \frac{1}{R_2 - R_1} \quad \text{for all } j \quad \text{by (2.6)}.$$

3. Function spaces associated with vector fields

In order to study weak solutions, we introduce a function space $M^k(\Omega)$ associated with X, which is an analogue of the usual Sobolev space. For $k \geq 1$, we define

(3.1) $$M^k(\Omega) := \left\{f \in L^2(\Omega); X^J f \in L^2(\Omega), \quad \forall |J| \leq k\right\},$$

where $J = (j_1, \ldots, j_s)$, $1 \leq j_l \leq m$, $|J| = s$, $X^J f = X_{j_1} \ldots X_{j_s} f$. This weighted Sobolev space $M^k(\Omega)$ is a separable Hilbert space. We denote by $M_0^k(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $M^k(\Omega)$. From Hörmander's condition, we have

(3.2) $$M_0^k(\Omega) \subset H^{\frac{k}{r}}(\Omega).$$

We also define Hölder spaces by

(3.3) $$S^{0,\alpha}(\Omega) := \left\{u \in L^\infty(\Omega); [u]_X^\alpha = \sup_{x,y\in\Omega} \frac{|u(x) - u(y)|}{\varrho(x,y)^\alpha} < +\infty\right\},$$

(3.4) $$S^{k,\alpha}(\Omega) := \left\{ u \in S^\alpha(\Omega); X^J \in S^k(\Omega), \quad \forall |J| \le k \right\}.$$

These non-isotropic Hölder spaces are Banach spaces and satisfy

(3.5) $$S^{kr,\alpha}(\Omega) \subset C^{k,\frac{\alpha}{r}}(\Omega).$$

We usually write S^k instead of $S^{k,0}$. These function spaces are well adapted to the operator H. Using the degenerate Riemannian structure of § 2, and the present function spaces, the Hörmander operator H enjoys many Laplacian type properties. We collect here those results that we shall exploit in the sequel.

Lemma 3.1 (Maximum principle of Bony [B]). *Let* $u \in S^{1,\alpha}(\Omega)$ *satisfy* $Hu \le 0$, $\alpha > 0$. *If there exists* $x^0 \in \Omega$ *with* $0 \le u(x^0) = \sup_\Omega u(x) < +\infty$, *then* $u \equiv u(x^0)$ *on* Ω. *In fact, the maximum principle also holds for weak solutions* $u \in M^1(\Omega)$ *of* $Hu \le 0$ *in the usual sense.*

Since the operator H is hypoelliptic, the maximum principle immediately yields the uniqueness of (weak) solutions for the Dirichlet problem on Ω.

Lemma 3.2.   (i) *(First Poincaré inequality on* Ω*)*

(3.6) $$\|\varphi\|_{L^2(\Omega)} \le C(\Omega) \|X\varphi\|_{L^2(\Omega)}, \quad \forall \varphi \in M_0^1(\Omega).$$

(ii) *(First Poincaré inequality on* $B_H(x_0, R)$ *for* R *small enough).*
      $\forall x_0 \in \Omega \; \exists R_0 > 0, C(R_0) > 0$ *such that* $\forall 0 < R \le R_0$

(3.7) $$\|\varphi\|_{L^2(B_H(x_0,R))} \le C(R_0)R \|X\varphi\|_{L^2(B_H(x_0,R))}, \quad \forall \varphi \in M_0^1(B_H(x_0,R))$$

(iii) $\forall x_0 \in \Omega \; \exists R_0 > 0$ *with* $B_H(x_0, 2R_0) \subset \Omega$, $C(R_0) > 0$, *such that for* $0 < R \le R_0$

(3.8) $$\int_{B_H(x_0,R)} |u - \bar{u}_R|^2 \le C(R_0)R^2 \int_{B_H(x_0,R)} |Xu|^2$$

   $\forall u \in M^1(B_H(x_0, R))$, *where*

$$\bar{u}_R = |B_H(x_0, R)|^{-1} \int_{B_H(x_0,R)} u(x)dx.$$

Proof. **(i)** follows from the strong maximum principle of Lemma 3.1.

(ii) For $x^0 \in \Omega$, Hörmander's condition implies that there exists j_0 such that $X_{j_0}(x_0) \ne 0$. We change the variables so that $X_{j_0} = \partial_{x_k}$ in $B_H(x_0, R) \subset B_\Delta(x_0, R)$. The classical Poincaré inequality then gives the results.

(iii) is a result of Jerison [Je]. We remark that, using the proof of [Je], we can also obtain

(3.9) $$\int_{T_R} |u - \bar{\bar{u}}_R|^2 \le CR^2 \int_{T_R} |Xu|^2$$

where $T_R = B_H(x_0, R) \setminus B_H(x_0, \frac{1}{2}R)$ and $\bar{\bar{u}}_R = |T_R|^{-1} \int_{T_R} u(x)dx$.   q.e.d.

The hypoellipticity of the operator H, and the maximum principle imply that there exists the Green function $G(x, y)$ of H on Ω. We have

Lemma 3.3 (see [NSW], [S]). $\forall K \subset\subset \Omega$, *we have* $\forall x, y \in K$,

(3.10) $$0 < G(x, y) \le C_K \varrho(x, y)^2 |B_H(x, \varrho(x, y))|^{-1},$$

(3.11) $$\left| X^J G(x, y) \right| \le C_{J,K} \varrho(x, y)^{2-|J|} |B_H(x, \varrho(x, y))|^{-1},$$

(3.12) $\qquad \varrho(x,y)^2 \, |B_H(x, \varrho(x,y))|^{-1} \leq C_K G(x,y) \quad for \; \varrho(x,y) \leq \delta.$

Associated with this metric we have also

Lemma 3.4 (see [X2]). *Let $K \subset\subset \Omega$ and put $K_\delta := \{y \in \Omega, \varrho(x,y) < \delta, x \in K\}$. Then, for $0 < \delta \leq \delta_0$ small enough $\exists \varphi_\delta \in C_0^\infty(K_\delta)$ such that $\varphi_\delta(x) = 1$, $x \in K$, and $|X^J \varphi_\delta| \leq C_J / \delta^{|J|}$, where C_J is independent of δ.*

Lemma 3.5 (see [X2]). *For all $\alpha > 0$ and $0 < R \leq R_0$,*

(3.13) $\qquad \fint_{B_H(x,R)} \varrho(x,y)^\alpha \, |B_H(x, \varrho(x,y))|^{-1} \, dy \leq C_{R_0, \alpha} R^\alpha.$

Using these results, we can obtain the optimal interior regularity results for the operator H.

Lemma 3.6 (see [RS], [X2]). *Suppose that $Hu = f$. Then, if $f \in M^k(\Omega)$, we have $u \in M^{k+2}(\Omega')$, and if $f \in S^{k,\alpha}(\Omega)$, we have $u \in S^{k+2,\alpha}(\Omega')$ for any $k \in \mathbb{N}$, $\alpha > 0$, and all $\Omega' \subset\subset \Omega$. We also have the a priori estimates*

(3.14) $\qquad \|\varphi u\|_{M^{k+2}(\Omega')} \; \leq \; C(\Omega') \left\{ \|\varphi_1 Hu\|_{M^k(\Omega)} + \|\varphi_1 u\|_{L^2(\Omega)} \right\},$

(3.15) $\qquad \|\varphi u\|_{S^{k+2,\alpha}(\Omega')} \; \leq \; C(\Omega') \left\{ \|\varphi_1 Hu\|_{S^{k,\alpha}(\Omega)} + \|\varphi_1 u\|_{L^\infty(\Omega)} \right\},$

for all $\varphi \in C_0^\infty(\Omega')$, where $\varphi_1 \in C_0^\infty(\Omega)$, $\varphi_1(x) = 1$, $x \in \Omega'$.

In the sequel, we shall also need some further Green function estimates which we now collect. They are similar to the estimates of Biroli-Mosco [BM], but for completeness, we provide proofs.

Lemma 3.7. *Let $G(x,y)$ be the Green function of H on Ω. For $\sigma > 0$ small enough, we define the mollified Green function by*

(3.16) $$G^\sigma(x,y) = \fint_{B(y,\sigma)} G(x,z) dz.$$

Then, $G^\sigma(\cdot, y) \in M_0^1 \cap L^\infty(\Omega, R)$, and

(3.17) $\qquad 0 \leq G^\sigma(x,y) \leq C \begin{cases} \varrho(x,y)^2 \, |B(x, \varrho(x,y))|^{-1}, & if \; \sigma < c_1 \varrho(x,y), \\ \sigma^2 \, |B(x, \sigma)|^{-1}, & if \; \sigma \geq c_1 \varrho(x,y), \end{cases}$

and

(3.18) $\qquad |X^x G^\sigma(x,y)| \leq C \begin{cases} \varrho(x,y) \, |B_H(x, \varrho(x,y))|^{-1}, & if \; \sigma < c_1 \varrho(x,y), \\ \sigma \, |B_H(x, \sigma)|^{-1}, & if \; \sigma \geq c_1 \varrho(x,y), \end{cases}$

and $\forall \xi \in M_0^1(\Omega)$, we have, if $B(y,\sigma) \subset \Omega$,

(3.19) $$\int_\Omega X\xi G^\sigma(\cdot, y) = \fint_{B(y,\sigma)} \xi(x) dx$$

and $\lim_{\sigma \to \infty} G^\sigma(x,y) = G(x,y)$.

Proof. (3.19) is evident. Using Lemma 3.3 and (2.1), (2.4), we get (3.17) and

$$
\begin{aligned}
|X^x G^\sigma(x,y)| \;&= \; \left| \fint_{B_\sigma(y)} X^x G(x,z) dz \right| \\
&\leq \; C \begin{cases} \varrho(x,y) \, |B_H(x, \varrho(x,y))|^{-1}, & \sigma \leq c_1 \varrho(x,y), \\ \delta \, |B_H(x, \delta)|^{-1}, & \sigma \geq c_1 \varrho(x,y). \end{cases}
\end{aligned}
$$

We have also proved $G^\sigma(\cdot, y) \in S^{1,\alpha}(\bar{\Omega}) \ \forall 0 < \alpha < 1$.   q.e.d.

Corollary 3.1. *If $G(x,y)$ is the Green function of H on Ω, then*

$$(3.20) \qquad 0 \le G(x,y) \le C\varrho(x,y)^2 \left| B(x, \varrho(x,y)) \right|^{-1} \quad \forall x, y \in \bar{\Omega}.$$

Proof. Denote by $\tilde{G}(x,y)$ the Green function of H on $\tilde{\Omega} \supset\supset \Omega$. Then $\tilde{G}^\sigma(\cdot, y) \in S^{1,\alpha}(\bar{\Omega})$. We now solve the following Dirichlet problem:

$$\begin{cases} HU_y^\sigma = 0, \\ U_y^\sigma|_{\partial\Omega} = \tilde{G}^\sigma(\cdot, y) \in C^0(\partial\Omega). \end{cases}$$

Then the theorem of Bony [B] gives the existence of a solution $U_y^\sigma \in C^0(\bar{\Omega})$. Set $G_y^\sigma(x) = U_y^\sigma(x) + \tilde{G}^\sigma(x,y)$; then $H(G_y^\sigma - \tilde{G}^\sigma(\cdot, y)) = 0$, and the maximum principle implies

$$G_y^\sigma(x) \le \tilde{G}^\sigma(x,y) \quad \forall x, y \in \bar{\Omega}.$$

(3.20) now follows by letting $\sigma \to 0$.   q.e.d.

4. DIRICHLET AND ENERGY FUNCTIONALS ASSOCIATED WITH A HÖRMANDER OPERATOR

Let N be a complete Riemannian manifold without boundary of dimension ≥ 2. We suppose that N can be covered by a single coordinate chart, so that we can define the weighted Sobolev space $M^1(\Omega, N)$ unambiguously with the help of this chart, i.e.

$$(4.1) \qquad f \in M^1(\Omega, N) \Leftrightarrow \psi \circ f \in M^1(\Omega, \mathbb{R}^\nu),$$

where $\psi : Y \to \mathbb{R}^\nu$ is a coordinate chart of class C^∞. Similarly, we define the spaces $M_0^1(\Omega, N)$ and $S^{k,\alpha}(\Omega, N)$. For $p \in N$ and $r > 0$, we put $B(p,r) = \{q \in N : d_N(p,q) \le r\}$, where the distance function d_N comes from the Riemannian metric. In intrinsic notation, the metric of N is denoted by $\langle \cdot, \cdot \rangle_N$. In local coordinates the coefficients of the metric of N are denoted by $(g_{\alpha\beta}(x))$, with $(g^{\alpha\beta}(x))$ being the inverse of $(g_{\alpha\beta})$, while $\Gamma_{\beta\gamma}^\alpha$ are the Christoffel symbols.

We consider maps

$$f : \left(\bar{\Omega}, d_\varepsilon \right) \to (N, d_N).$$

Since $(\bar{\Omega}, d_\varepsilon)$, (N, d_N) are Riemannian manifolds, with metric tensors (a_{ij}^ε) and $(g_{\alpha\beta})$, if $f : \bar{\Omega} \to N$ is a C^1 map, we can define the energy density

$$\begin{aligned} e_\varepsilon(f) &= \frac{1}{2} \left(a^{ij}(x) + \varepsilon\delta_{ij} \right) g_{\alpha\beta}(f) \frac{\partial f^\alpha}{\partial x_i} \frac{\partial f^\beta}{\partial x_j} \\ (4.2) \qquad &= \frac{1}{2} g_{\alpha\beta}(f) X_j f^\alpha X_j f^\beta + \frac{\varepsilon}{2} g_{\alpha\beta}(f) \frac{\partial f^\alpha}{\partial x_j} \frac{\partial f^\beta}{\partial x_j}. \end{aligned}$$

Then the energy of f is simply

$$(4.3) \qquad E_\varepsilon(f) = \int_\Omega e_\varepsilon(f) d\Omega$$

with $d\Omega = dx$, the Lebesgue measure. For $f \in H^1(\Omega, N)$,

$$\lim_{\varepsilon \to 0} E_\varepsilon(f) = \int_\Omega e(f) dx,$$

where

(4.4) $$e(f) = \frac{1}{2} g_{\alpha\beta}(f) X_j f^\alpha X_j f^\beta.$$

So we define the energy of a map $f : (\bar{\Omega}, \varrho) \to (N, d_N)$ by

(4.5) $$E(f) = \int_\Omega e(f) dx.$$

The Euler-Lagrange equation for this energy functional can be written in the form

(4.6) $$H u^\alpha + \Gamma^\alpha_{\beta\gamma}(u) X_j u^\beta X_j u^\gamma = 0, \qquad 1 \le \alpha \le \nu.$$

We abbreviate (4.6) as $H_N u = 0$ (thus $H_\mathbb{R} = H$).

We can now state our existence result for a solution of the Dirichlet problem for (4.6):

Main Theorem. *Let κ^2 be an upper bound for the sectional curvature of N, and suppose that $p \in N$ and $\mu < \min(\pi/2\kappa, i(p))$, where $i(p)$ is the injectivity radius of $p \in N$. Let $\varphi \in C^0(\bar{\Omega}, N) \cap M^1(\Omega, N)$ satisfy*

(4.7) $$\varphi(\bar{\Omega}) \subset B(p, \mu).$$

Then there exists a unique map $u \in M^1(\Omega, N) \cap L^\infty(\Omega, N)$ with $u|_{\partial\Omega} = \varphi$, $u(\bar{\Omega}) \subset B(p, \mu)$, minimizing E among all such maps, and this map u is a weak solution of (4.6). Moreover, u enjoys the same interior regularity properties as solutions of linear hypoelliptic systems, and if $\partial\Omega$ is smooth and noncharacteristic for $X_1, \dots, X_m$, and if φ is smooth, one gets the corresponding boundary regularity of u.

5. Existence of weak solutions

The weighted Sobolev space $M^1(\Omega, \mathbb{R}^\nu)$ carries a scalar product with associated norm

(5.1) $$||u||^2_{M^1} = \int_\Omega |u|^2 \, dx + \int_\Omega |Xu|^2 \, dx.$$

We fix $p \in N$ as in Theorem 1; for $\mu' < \min(\pi/2\kappa, i(p))$, we put

$$B_{\mu'} := \left\{ u \in M^1(\Omega, N); \sup_\Omega d_N(u(x), p) \le \mu' \right\}.$$

In geodesic normal coordinates with center p, $B_{\mu'}$ can be identified with

$$\left\{ u \in M^1 \cap L^\infty(\Omega, \mathbb{R}^\nu), \sup_\Omega |u| \le \mu' \right\}.$$

We also put

$$B_{\mu'}(\Phi) := \left\{ u \in B_{\mu'} : u - \Phi \in M^1_0(\Omega, N) \right\} \quad \text{for given } \Phi \in B_{\mu'}.$$

Since E is a Dirichlet integral, E is lower semicontinuous on $B_{\mu'}$, i.e. if $\{u_k\} \subset B_{\mu'}$ converges weakly to u in $M^1(\Omega, \mathbb{R}^\nu)$, then

(5.2) $$\liminf_{k \to \infty} E(u_k) \ge E(u).$$

The proof can be carried out with the methods of [G].

Proposition 5.1. *Let $\mu' < \min(\pi/2\kappa, i(p))$. For every $\Phi \in B_{\mu'}$, there exists a solution of the variational problem*

(5.3) $$E(u) \to \inf_{B_{\mu'}(\Phi)} E(v).$$

Proof. The above infimum is finite since $E(\Phi) < \infty$. Since $B(p, \mu')$ is compact and the metric tensor $(g_{\alpha\beta})$ is positive definite, there exists $\lambda > 0$ with

$$E(v) \geq \lambda \int_\Omega |Xv|^2 \, dx \quad \text{for all } v \in B_{\mu'}.$$

Consequently, for all $v \in B_{\mu'}$,

$$\|v\|_{M^1}^2 \leq \text{const} \left\{ E(v) + (\mu')^2 \right\}.$$

Thus, a minimizing sequence is bounded in $M^1(\Omega, \mathbb{R}^\nu)$, and we may assume that it converges weakly to some $u \in B_{\mu'}(\Phi)$. By lower semicontinuity of E, we must have

$$E(u) = \inf_{B_{\mu'}(\Phi)} E(v),$$

so that u is the desired minimizer. q.e.d.

Remark. Using convexity properties of $B(p, \mu')$ (see § 6 below), one may even construct a minimizing sequence that converges strongly in M^1, thus bypassing the issue of weak lower semicontinuity of E.

In the sequel, we shall need the following chain rule:

Lemma 5.1. *Let $u \in C^2(\Omega, N)$, $f : N \to \mathbb{R}$, N a Riemannian manifold. Then*

$$(5.4) \qquad (-H)(f \circ u) = D^2 f(X_j u, X_j u) + \langle (\text{grad} \, f) \circ u, (-H_N) u \rangle_N,$$

where $D^2 f$ is the (Riemannian) Hessian of f and $\langle \cdot, \cdot \rangle_N$ the metric of N. In particular, if $H_N u = 0$, then

$$(5.5) \qquad\qquad (-H)(f \circ u) = D^2 f(X_j u, X_j u).$$

If in addition, f is strictly convex, with smallest eigenvalue $\lambda > 0$ of its Hessian, then

$$(5.6) \qquad\qquad (-H)(f \circ u) \geq \lambda |Xu|^2.$$

If we only assume $u \in M^1(\Omega, N)$, then under the preceding assumptions, (5.6) holds in the weak sense.

Proof. See the computations in [J1]. q.e.d.

We shall now deduce the following maximum principle:

Lemma 5.2. *Under the assumptions of Proposition 5.1, suppose $\Phi \in B_\mu$ for some $\mu < \mu'$. Then also the minimizer u of Proposition 5.1 is contained in B_μ, i.e.*

$$(5.7) \qquad\qquad \sup_\Omega d_N(u(x), p) \leq \mu.$$

Proof. We consider the function

$$f : N \to \mathbb{R},$$

$$f(q) = d_N^2(q, p).$$

f is smooth on $B(p, \mu')$, since $\mu < i(p)$, and f is strictly convex on $B(p, \mu')$, since $\mu < \pi/2\kappa$ (see e.g. [J1] for details). This implies that the projection

$$\pi : B(p, \mu') \to B(p, \mu)$$

along radial geodesics is energy decreasing, i.e.

$$(5.8) \qquad E(\pi \circ u) < E(u) \quad \text{unless } u(x) \subset B(p, \mu) \quad \text{for almost all } x \in \Omega$$

(again see [J1]). Since by assumption $\pi \circ \Phi = \Phi$, we have $\pi \circ u \in B_\mu(\Phi)$, and (5.8) and the minimizing property of u therefore imply the claim. q.e.d.

We may now obtain the existence part of the Main Theorem, which we formulate as

Theorem 1. *Let $p \in N$, $\mu < \min(i(p), \pi/2\kappa)$, and suppose $\Phi \in M^1(\Omega, N)$ satisfies $\Phi(\bar\Omega) \subset B(p, \mu)$. Then there exists a weak solution u of*

$$(5.9) \qquad \int_\Omega X_j u^\alpha X_j \varphi^\alpha - \int_\Omega \Gamma^\alpha_{\beta\gamma}(u) X_j u^\beta X_j u^\gamma \varphi^\alpha = 0$$

for all $\varphi \in M_0^1(\Omega, \mathbb{R}^\nu) \cap L^\infty(\Omega, \mathbb{R}^\nu)$,

$$(5.10) \qquad\qquad u - \Phi \in M_0^1(\Omega, N),$$

and u satisfies

$$(5.11) \qquad\qquad u(\bar\Omega) \subset B(p, \mu).$$

Proof. We minimize E in $B_{\mu'}(\Phi)$ for some μ' with $\mu < \mu' < \min(i(p), \pi/2\kappa)$. The solution u exists by Proposition 5.1, and Lemma 5.2 implies $u \in B_\mu(\Phi)$, i.e. (5.11). Therefore, u lies in the interior of $B_{\mu'}(\Phi)$, and we may perform variations $u + \varepsilon\varphi$ for small enough $|\varepsilon|$ inside $B_{\mu'}(\Phi)$ and exploit the minimizing property of u to arrive at (5.9). q.e.d.

6. Interior continuity of weak solutions

Suppose u is a weak solution of $H_N u = 0$ ($u : M \to N$), $f : u(M)(\subset N) \to \mathbb{R}$ strictly convex, with $\lambda :=$ smallest eigenvalue of $D^2 f(> 0)$. Let

$$h := f \circ u.$$

Then by (5.6)

$$(6.1) \qquad\qquad (-H)h \geq \lambda \sum_j |X_j u|^2 \quad \text{weakly } (=: \lambda e(u)).$$

We let G_{x_0} be the Green function for H on Ω with singularity at x_0.

For $0 < \varrho \leq \text{dist}(x, \partial\Omega)$, we let

$$g_\varrho(x) := \min\left(G_{x_0}(x) - \varrho^2 \, |B(x_0, \varrho)|^{-1}, \left(\frac{\varrho}{2}\right)^2 |B(x_0, \varrho/2)|^{-1} - \varrho^2 \, |B(x_0, \varrho)|^{-1} \right)$$

and

$$\Gamma_\varrho := \{x : g_\varrho(x) = 0\}.$$

Then, for $0 < \varrho' = \frac{\varrho''}{2} < \varrho'' < \mathrm{dist}(x, \partial\Omega)$,

$$
\begin{aligned}
0 \;\; &\leq \;\; \lambda \int_{\underset{0 \leq \varrho \leq \varrho''}{\bigcup \Gamma_\varrho}} g_{\varrho''} e(u) \\
&\leq \;\; \int_{\underset{0 \leq \varrho \leq \varrho''}{\bigcup \Gamma_\varrho}} g_{\varrho''}(-H)h \\
&= \;\; -\int_{\underset{\varrho' \leq \varrho \leq \varrho''}{\bigcup \Gamma_\varrho}} X_j G_{x_0} X_j h \\
&= \;\; -\int_{\underset{0 \leq \varrho \leq \varrho''}{\bigcup \Gamma_\varrho}} X_j G_{x_0} X_j h + \int_{\underset{0 \leq \varrho \leq \varrho'}{\bigcup \Gamma_\varrho}} X_j G_{x_0} X_j h
\end{aligned}
$$

(for the existence of these integrals, one uses the mollified Green's function as in Lemma 3.7 and lets the mollification parameter tend to 0)

$$
(6.2) \qquad\qquad =: \;\; \mu(\varrho'') - \mu(\varrho').
$$

Now let $0 < R \leq \mathrm{dist}(x, \partial\Omega)$ and

$$
R_i := R \cdot 2^{-i}.
$$

Then from (6.2)

$$
\begin{aligned}
\int_{\underset{0 \leq \varrho \leq R}{\bigcup \Gamma_\varrho}} G_{x_0} e(u) \;\; &\leq \;\; c \sum_{i=0}^{\infty} \int_{\underset{0 \leq \varrho \leq R_i}{\bigcup \Gamma_\varrho}} g_{R_i} e(u) \\
&\leq \;\; \frac{c}{\lambda} \sum_{i=0}^{\infty} \left(\mu(R_i) - \mu(R_{i+1}) \right) \\
(6.3) \qquad &\leq \;\; \frac{c}{\lambda} \mu(R).
\end{aligned}
$$

In particular,

$$
(6.4) \qquad\qquad \int_{B_H(x_0, R)} G_{x_0} e(u) < \infty,
$$

and given $\delta > 0$, for sufficiently large j, since $\lambda > 0$,

$$
(6.5) \qquad\qquad \int_{B_H(x_0, R_j)} G_{x_0} e(u) < \delta.
$$

(For the telescoping trick, see Giaquinta-Giusti [GG] and Giaquinta-Hildebrandt [GH].)

If u is a weak solution of $H_N u = 0$, then

$$
(6.6) \qquad\qquad \int \langle X_j u, X_j \varphi \rangle_N = 0 \quad \text{for all } M_0^1\text{-sections } \varphi \text{ of } u^{-1}TN.
$$

We now choose

$$
\varphi = \eta \cdot G_y \cdot (Df) \circ u,
$$

where f is a strictly convex C^2 function on $u(M)$, η is a cut-off function, $\eta \in \mathrm{Lip}(B_H(x_0, 2R_0))$, $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B_H(x_0, R_0)$, $|X_j \eta| \leq c_1/R_0$, $\mathrm{supp}\, \eta \subset\subset B_H(x_0, 2R_0)$ (see the discussion at the end of § 2), and where G_y is Green's function on Ω with singularity this time at $y \in B_H(x_0, R_0/2)$, with $B_H(x_0, R_0) \subset \Omega$. Since G_y is unbounded, φ as it stands is not an admissible test function, but replacing

G_y by a mollification or a cut-off, one easily sees by an approximation argument using Lemma 3.7 that the subsequent computations are valid. This choice of φ is patterned after Hildebrandt-Widman [HW], and so are some of the subsequent estimates. With this choice of φ, (6.6) becomes

$$
\begin{aligned}
\int_{B_H(x_0, 2R_0)} X_j G_y X_j(\eta f(u)) \quad = \quad & -\int_{B_H(x_0, 2R_0)} \eta G_y D^2 f(X_j u, X_j u) \\
& -\int_{B_H(x_0, 2R_0)} G_y X_j \eta X_j f(u) \\
(6.7) \qquad\qquad & +\int_{B_H(x_0, 2R_0)} f(u) X_j G_y X_j \eta.
\end{aligned}
$$

We write (6.7) as

$$(6.8) \qquad\qquad I = II + III + IV.$$

Since $\eta = 0$ on $\partial B_H(x_0, 2R_0)$ and $\eta(x_0) = 1$, we have

$$(6.9) \qquad\qquad I = f(u(y)).$$

Since η and G_y are nonnegative and f is convex,

$$(6.10) \qquad\qquad II \le 0.$$

Let $c_2 := \sup_{u(M)} |Df|$. Then

$$
\begin{aligned}
III \quad \le \quad & \frac{c_1 c_2}{R_0} \int_{B_H(x_0, 2R_0) \setminus B_H(x_0, R_0)} G_y \, |X_j u|_N \\
\le \quad & \frac{c_1 c_2}{R_0} \left\{ \int_{B_H(x_0, 2R_0) \setminus B_H(x_0, R_0)} G_y \right\}^{\frac{1}{2}} \left\{ \int_{B_H(x_0, 2R_0) \setminus B_H(x_0, R_0)} G_y e(u) \right\}^{\frac{1}{2}}.
\end{aligned}
$$
(6.11)

Now from (3.10)

$$
\int_{B_H(x_0, 2R_0) \setminus B_H(x_0, R_0)} G_y \le c_3 R_0^2,
$$

and therefore, recalling (6.5), by choosing R_0 sufficiently small, depending on a given $\varepsilon > 0$, we may achieve

$$(6.12) \qquad\qquad III \le \varepsilon,$$

because the Green functions with singularities at x_0 and at $y \in B_H(x_0, R_0/2)$ are comparable on the annulus $B_H(x_0, 2R_0) \setminus B_H(x_0, R_0)$.

We now estimate IV. We let u_{R_0} be the mean value of u on

$$B_H(x_0, 2R_0) \setminus B_H(x_0, R_0).$$

This mean value can be defined with the help of convex geometry on the image of u, for example if the image has nonpositive curvature or if the image is contained in a convex ball as described below. Alternatively, one might define it with the help of local coordinates. The different versions will only affect the constants that appear in the Poincaré inequality that will be used in a moment.

We also make the further assumptions that f has a minimum at u_{R_0} and that

$$(6.13) \qquad\qquad f(u_{R_0}) = 0.$$

Then

$$IV = \int_{B_H(x_0,2R_0)\setminus B_H(x_0,R_0)} f(u) X_j G_y X_j \eta$$

$$\leq \frac{c_4}{|B_H(x_0,2R_0)|} \int_{B_H(x_0,2R_0)\setminus B_H(x_0,R_0)} f(u) \quad \text{by (3.11)}$$

$$\leq \frac{c_5}{|B_H(x_0,2R_0)|^{\frac{1}{2}}} \sup_{u(M)} |Df| \left\{ \int_{B_H(x_0,2R_0)\setminus B_H(x_0,R_0)} d_N(u, u_{R_0})^2 \right\}^{\frac{1}{2}}$$

$$(6.14) \qquad \leq c_6 \left\{ \frac{R_0^2}{|B_H(x_0,2R_0)|} \int_{B_H(x_0,2R_0)\setminus B_H(x_0,R_0)} e(u) \right\}^{\frac{1}{2}}$$

by the Poincaré inequality (3.8),

where $\sup |Df|$ now has been incorporated into the constant c_6. (6.5) now can be used as before to obtain

$$(6.15) \qquad\qquad\qquad\qquad IV \leq \varepsilon$$

for given $\varepsilon > 0$ and sufficiently small R_0 depending on ε.

Altogether, (6.7)–(6.15) yield

$$(6.16) \qquad\qquad f(u(y)) \leq 2\varepsilon \quad \text{for all } y \in B\left(x_0, \frac{R_0}{2}\right),$$

where R_0 depends on ε. We may now deduce the interior regularity part of the Main Theorem.

Theorem 2. *Let N be a Riemannian manifold, $u : M \to N$ a bounded weak solution of*

$$H_N u = 0.$$

Suppose

$$(6.17) \qquad\qquad u(M) \subset B(p,\mu) := \{q \in N : d_N(p,q) \leq \mu\}$$

for some $p \in N$ with

$$(6.18) \qquad\qquad \mu < \min\left(\frac{\pi}{2\kappa}, i(p)\right),$$

where the sectional curvature K of N satisfies $K \leq \kappa^2$, and where $i(p)$ is the injectivity radius of p. (A ball $B(p,\mu)$ satisfying (6.18) will be called a convex ball.)

Then u is continuous.

Proof. Building upon (6.16), this may be shown by an iteration argument first developed by Wiegner [Wi] and refined by Hildebrandt-Widman [HW], Hildebrandt-Kaul-Widman [HKW], Hildebrandt-Jost-Widman [HJW]. However, the reasoning may be substantially simplified by employing the result of Kendall [K] that for each $q \in B(p,\mu)$ (where $B(p,\mu)$ is a convex ball), there exists a strictly convex function

$$f = f_q : B(p,\mu) \to \mathbb{R}$$

with its unique minimum at x_0. In fact, inspired by [JäK] we may take

$$f_q(y) = \frac{1 - \cos(\kappa \operatorname{dist}(q, y))}{\cos(\kappa \operatorname{dist}(p, y))},$$

as shown by Kendall.

Putting $q = u(x_0)$ and $f = f_{u(x_0)}$ in (6.16) immediately yields continuity of u at x_0, an arbitrary point of M.   q.e.d.

Theorem 2 in particular applies in the case where the domain is simply connected and N has nonpositive sectional curvature, because we may lift to universal covers, and on the universal cover of a manifold of nonpositive sectional curvature the injectivity radius is always infinite, so that (6.18) holds for any positive μ.

7. INTERIOR HÖLDER CONTINUITY

We return to our decomposition (6.7), (6.8). As in the proof of Theorem 2, by using Kendall's result [K], we may assume that the smallest eigenvalue of $D^2 f$ is $\geq \lambda > 0$. We therefore have, with $y = x_0$ and R instead of R_0,

$$(7.1) \qquad \lambda \int_{B_H(x_0, R)} G_{x_0} |Xu|^2 \leq \int_{B_H(x_0, 2R)} \eta G_{x_0} D^2 f(X_j u, X_j u).$$

We may further assume that $f(u(x_0)) = 0$, so that $I = 0$ for $y = x_0$ (cf. (6.9)). Using (6.11), (6.14) and Lemma 3.3, we obtain from (6.8)

$$(7.2) \qquad \int_{B_H(x_0, 2R)} \eta G_{x_0} D^2 f(X_j u, X_j u) \leq c_0 \int_{T_R} G_{x_0} |Xu|^2,$$

with the abbreviation

$$T_R := B_H(x_0, 2R) \setminus B_H(x_0, R).$$

(7.1) and (7.2) imply

$$(7.3) \qquad \int_{B_H(x_0, R)} G_{x_0} |Xu|^2 \leq c_1 \int_{T_R} G_{x_0} |Xu|^2.$$

We may then use Widman's hole filling, i.e. add

$$c_1 \int_{B_H(x_0, R)} |Xu|^2 G_{x_0}$$

to both sides of (7.3), to obtain

$$(7.4) \qquad \int_{B_H(x_0, R)} |Xu|^2 G_{x_0} \leq \vartheta \int_{B_H(x_0, 2R)} |Xu|^2 G_{x_0}$$

for some $0 < \vartheta < 1$ ($\vartheta = c_1/(1 + c_1)$). Using the standard iteration method originally introduced by de Giorgi, (7.3) yields

$$(7.5) \qquad \int_{B_H(x_0, r)} |Xu|^2 G_{x_0} \leq c_2 r^\alpha R_0^{-\alpha} \int_{B_H(x_0, R_0)} |Xu|^2 G_{x_0}$$

for $0 < r < \frac{1}{2} R_0$ and some positive α. Recalling (6.4), this means that

$$(7.6) \qquad \int_{B_H(x_0, r)} |Xu|^2 G_{x_0} \leq c(R_0) r^\alpha.$$

If we then use the estimates (6.11) and (6.14) with $r/2$ in place of R_0 and combine them with (7.5), we obtain for $y \in B_H(x_0, r/4)$

$$(7.7) \qquad f(u(y)) \leq c_3 r^\alpha,$$

and thus Hölder continuity by the properties of f. We thus have shown

Theorem 3. *Under the assumptions of Theorem 2, u is Hölder continuous in the interior of Ω.*

The proof again essentially follows the lines of [HW] and [HKW]. See also [J1].

8. Regularity up to the boundary

We suppose now that the boundary of Ω is regular in the following sense: $\exists c > 0$, $\exists R_0 > 0$ such that for any $x_0 \in \partial\Omega$, putting $D(x_0, R) := \Omega \cap B_H(x_0, R)$, we have

$$(8.1) \qquad |D(x_0, R)| \geq C\, |B_H(x_0, R)| \quad \forall 0 < R \leq R_0.$$

Near the boundary, for $R > 0$ small enough, we also have the following first Poincaré inequality:

$$(8.2) \qquad \int_{D(x_0, R)} |\varphi|^2\, dx \leq CR^2 \int_{D(x_0, R)} |X\varphi|^2\, dx$$

for any $\varphi \in M_0^1(\Omega)$. Using the fact that $\varphi|_{\partial\Omega \cap B_H(x_0, R)} = 0$, the proof is classical.

Theorem 4. *Let $\varphi \in S^1(\partial\Omega, N)$ and assume that $\partial\Omega$ is noncharacteristic for $X_1, \dots, X_m$. Suppose that under the assumptions of the preceding theorems, u solves the Dirichlet problem*

$$\begin{aligned} H_N u &= 0, \\ (8.3) \qquad u|_{\partial\Omega} &= \varphi. \end{aligned}$$

Then $u \in S^\alpha(\bar\Omega, N)$ for some $0 < \alpha < 1$.

Proof. By Theorem 3, it suffices to prove regularity near the boundary. Let $x_0 \in \partial\Omega$, and let f be a nonnegative strictly convex function on $B(p, \mu)$ with

$$(8.4) \qquad f(\varphi(x_0)) = 0,$$

i.e. $\varphi(x_0)$ is the unique minimum of f. As explained above, such a function was constructed by Kendall [K]. In order to have a noncharacteristic boundary, we may replace $D(x_0, R)$ by some $\tilde{D}_R$ with

$$D(x_0, R/2) \subset \tilde{D}_R \subset D(x_0, 3R/2)$$

for which $\partial\tilde{D}$ is noncharacteristic for $X_1, \dots, X_m$. We then solve the Dirichlet problem

$$\begin{aligned} H w_R &= 0 \quad \text{in } \tilde{D}_R, \\ (8.5) \qquad w_R|_{\partial\tilde{D}_R} &= f \circ u|_{\partial\tilde{D}_R}. \end{aligned}$$

The linear regularity theory implies that w_R is S^β near x_0, since $f \circ \varphi$ is S^1 on the boundary; more precisely,

$$(8.6) \qquad w_R(y) \leq w_R(x_0) + c d_H^\beta(x_0, y) \quad \text{for } y \in \tilde{D}_{\frac{R}{2}},$$

for some $0 < \beta < 1$. (In case φ is only assumed to be continuous, for all $\varepsilon > 0$ and $R > 0$ (sufficiently small), there exists r with

$$(8.7) \qquad w_R(y) \leq w_R(x_0) + \varepsilon \quad \text{for all } y \in D(x_0, r).)$$

By strict convexity of f again, we have

$$(8.8) \qquad H(f \circ u) \geq \lambda |Xu|^2 \geq 0 \quad \text{for some } \lambda > 0 \quad (\text{see } (5.6)).$$

Thus $H(f \circ u - w_R) \geq 0$, and (8.4)–(8.6) and the maximum principle yield

$$f(u(y)) - f(\varphi(x_0)) \leq cd_H^\beta(x_0, y),$$

hence Hölder continuity of u, since f is smooth and strictly convex with a unique minimum at $\varphi(x_0)$.   q.e.d.

Note. If we only assume that φ is continuous, we get continuity of u on $\bar\Omega$ by using (8.7) in place of (8.6).

9. Higher regularity

Theorem 5. *Under the assumptions of the main theorem, $u \in S^1(\Omega, N)$. If in addition $\varphi \in S^{1,\alpha}(\bar\Omega, N)$ and $\partial\Omega$ is sufficiently regular, then $u \in S^1(\bar\Omega, N)$.*

We only outline the proof, since in the preceding sections the reader has already seen sufficiently many instances of adapting known proofs in the elliptic case to the present hypoelliptic case.

Using (3.11), it is not hard to adapt the interior gradient bound of [JK] (see also [J1] and [JY]) to the present situation. Since we do not know yet, however, that u is of class S^1, we need to combine this estimate with an approximation argument. For that purpose, we approximate the metric d_H on Ω by Riemannian metrics d_ε as described in § 2. We then solve the corresponding Dirichlet problem for the operators H_ε and obtain solutions u_ε. These are known to be regular by [JY], and we get appropriate bounds independently of ε to allow us to obtain the desired bound for $u = \lim_{\varepsilon \to 0} u_\varepsilon$ (note that the solution of the Dirichlet problem is always unique under our assumptions on the image by [JäK]). In order to get these bounds, one needs to use the analogue of (3.11) for positive ε, with a constant independent of (sufficiently small) ε.

Similarly, the boundary gradient estimate of [JK] (again, see also [J1] and [JY]) extends in an even simpler manner.

Thus, one readily shows Theorem 5.

Theorem 5 implies that the nonlinear term in our system $H_N u = 0$ is bounded. One may therefore utilize the regularity theory of linear hypoelliptic PDE (see [X2]) and the standard bootstrap argument to get higher regularity.

This, together with Theorems 1–5 and the uniqueness result of [JäK] (see [J1] for a somewhat simplified proof), concludes the proof of the Main Theorem.

For a general higher regularity result, we also refer to [XZ].

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Institut für Mathematik, Ruhr-Universität Bochum, 44780 Bochum, Germany
Current address: Max-Planck-Institut für Mathematik in den Naturwissenschaften, Inselstr.
22-26, D-04103 Leipzig, Germany
E-mail address: `jost@mis.mpg.de`

Department of Mathematics, Wuhan University, 430072 Wuhan, China
Current address: Université de Rouen, Analyse et modèles stochastiques, UPRAS-A 6085,
UFR des Sciences, Mathématiques, F-76821 Mont St. Aignan, France
E-mail address: `chao-jiang.xu@univ-rouen.fr`